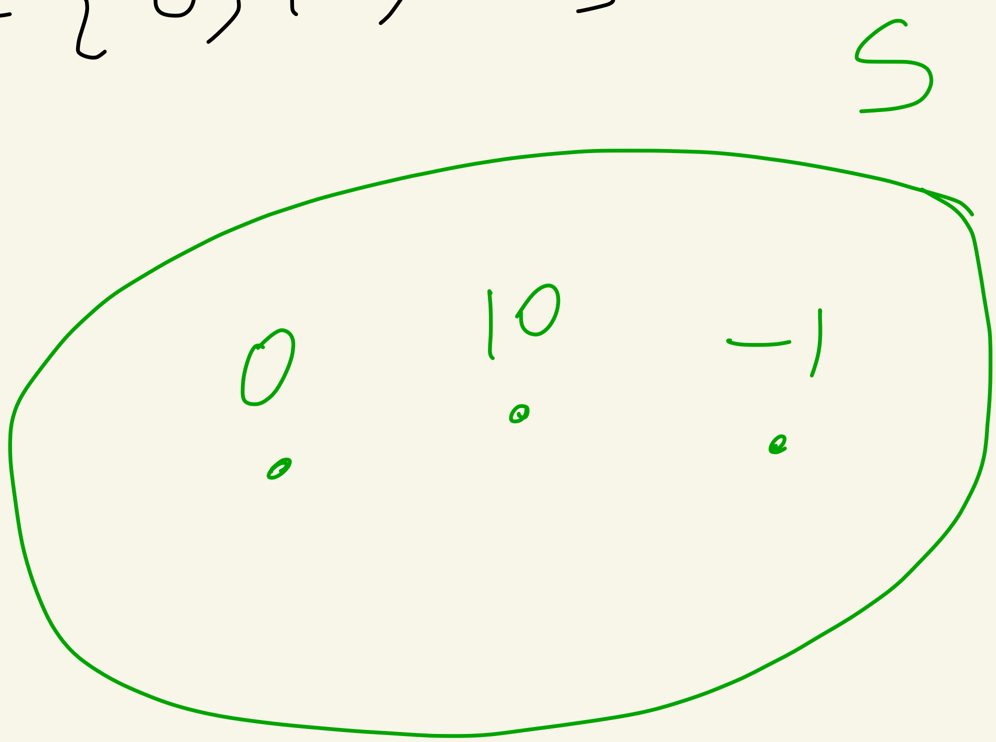
Topic 1 - Sets

Def: A __set__ is a collection of __elements__ (or objects).

If x is an element of a set S, then we write $\underbrace{x \in S}$.
read: "x is in S"

If x is not an element of a set S, then we write $\underbrace{x \notin S}$.
read: "x is not in S"

Ex: $S = \{0, 10, -1\}$



$0 \in S$

$10 \in S$

$-1 \in S$

$2 \notin S$

$-5 \notin S$

Note: There is no ordering on the elements of a set.

Thus,

$$\{0, 10, -1\} = \{10, -1, 0\}$$

for example.

Also, sets cannot have duplicate entries.

For example you can't have $\{1, 0, 1\}$ as a set.

Ex:

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \dots\}$ ← set of natural numbers

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ ← set of integers

---

general way to describe a set

$\{\text{description of element} \mid \text{condition on element to be in the set}\}$

read: "where"
"such that"
"given"

set of rational #s

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \text{ are integers and } q \neq 0 \right\}$$

$$= \left\{ \frac{1}{2}, \frac{2}{7}, \frac{-1}{23}, \frac{1}{137}, \frac{5}{1}, \dots \right\}$$

Ex:

$5 \in \mathbb{Q}$ $\quad\quad$ $\sqrt{2} \notin \mathbb{Q}$

$5 \in \mathbb{Z}$

$\frac{1}{2} \notin \mathbb{Z}$

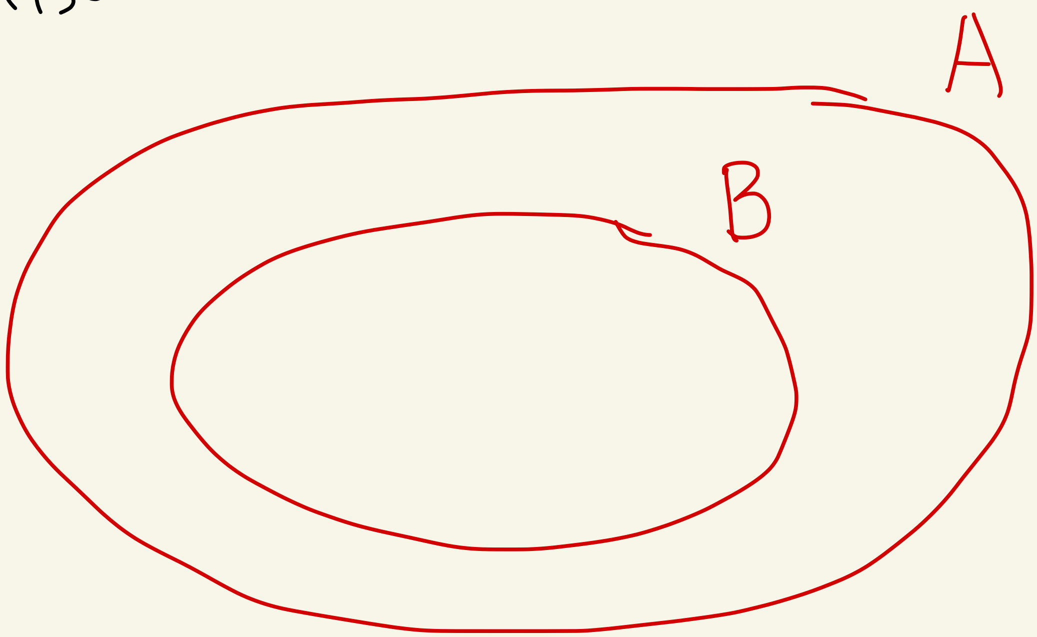
$$\mathbb{R} = \{ x \mid x \text{ has a decimal expansion} \}$$

$$= \{ 1, \frac{-5}{2}, \sqrt{2}, \pi, \dots \}$$

set of real numbers

1.0 $\quad$ -2.5 $\quad$ 1.414... $\quad$ 3.14159...

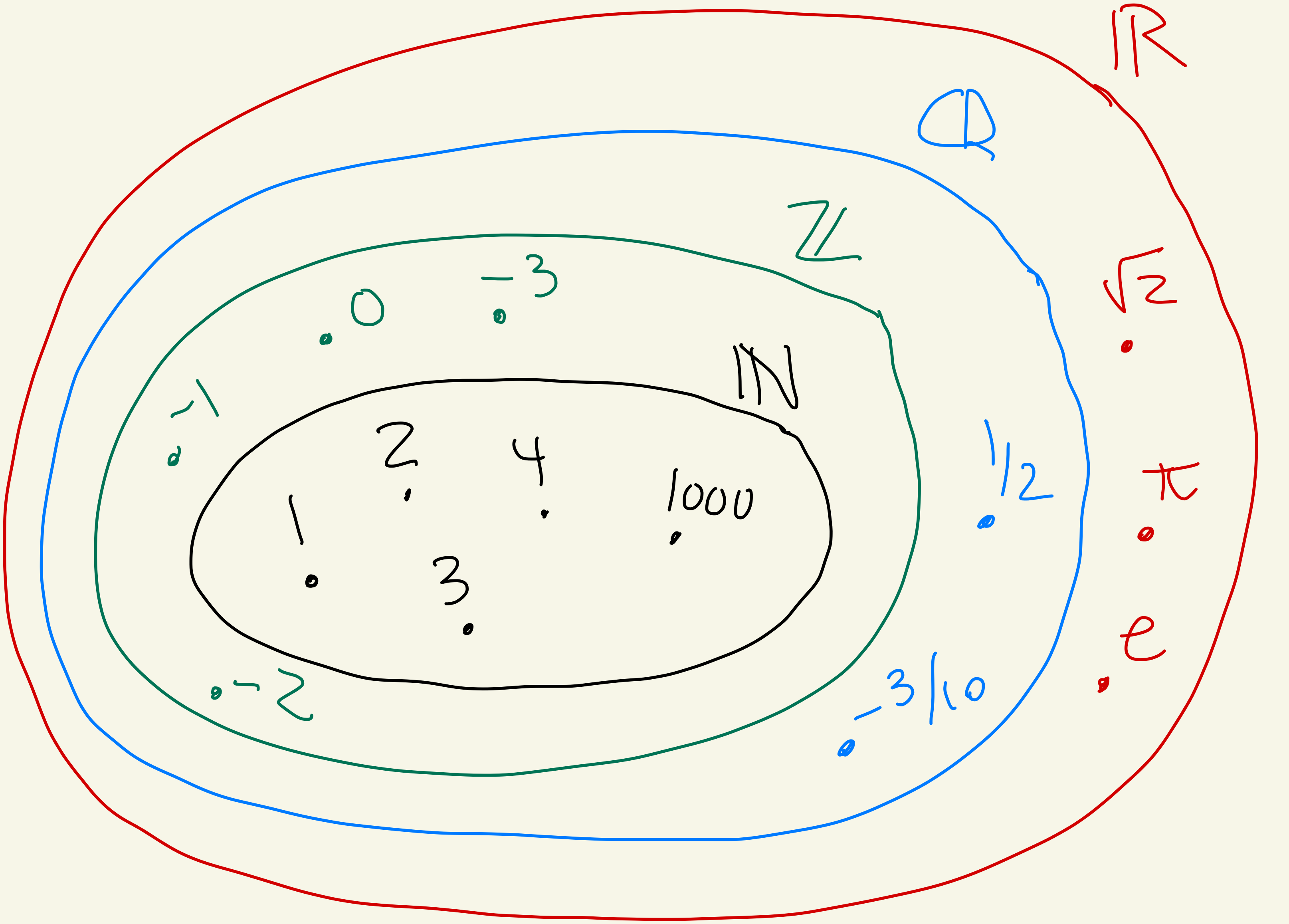
Def: Let A and B be sets. We say that B is a subset of A, and write $B \subseteq A$, if every element of B is also an element of A.



Some people write $B \subset A$ for subset

$\boxed{\text{Ex:}}$

$$\left.\begin{array}{l} \mathbb{N} \subseteq \mathbb{Z} \\ \mathbb{Z} \subseteq \mathbb{Q} \\ \mathbb{Q} \subseteq \mathbb{R} \end{array}\right\} \quad \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$



Ex: Let

$$A = \{12n \mid n \in \mathbb{Z}\}$$
$$B = \{3k \mid k \in \mathbb{Z}\}.$$

Then,

$$A = \{\ldots, 12(-3), 12(-2), 12(-1), 12(0), 12(1), 12(2), 12(3), \ldots\}$$
$$= \{\ldots, -36, -24, -12, 0, 12, 24, 36, \ldots\}$$

and

$$B = \{\ldots, 3(-3), 3(-2), 3(-1), 3(0), 3(1), 3(2), 3(3), \ldots\}$$
$$= \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}.$$

It seems that $A \subseteq B$.

Let's prove it formally.

$\downarrow$

<u>Technique:</u> To show that $A \subseteq B$ one way is to pick some $x \in A$ and then derive that $x \in B$.

<u>Ex:</u> Show that

$$\{12n \mid n \in \mathbb{Z}\} \subseteq \{3k \mid k \in \mathbb{Z}\}$$

<u>proof:</u>

Let $x \in \{12n \mid n \in \mathbb{Z}\}$.

Then, $x = 12n$ where $n \in \mathbb{Z}$.

Hence, $x = 3(4n)$.

Let $k = 4n$.

$4n \in \mathbb{Z}$ because $4, n \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under multiplication

So, $x = 3k$.

Thus, $x \in \{3k \mid k \in \mathbb{Z}\}$.

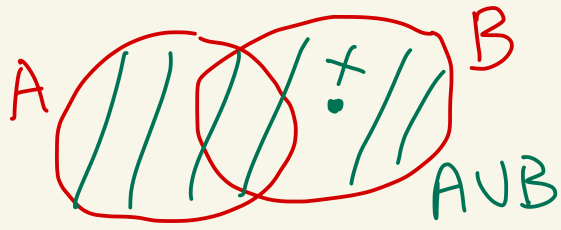
Therefore, $\{12n \mid n \in \mathbb{Z}\} \subseteq \{3k \mid k \in \mathbb{Z}\}$

$\blacksquare$

<u>Def:</u> Let A and B be sets.

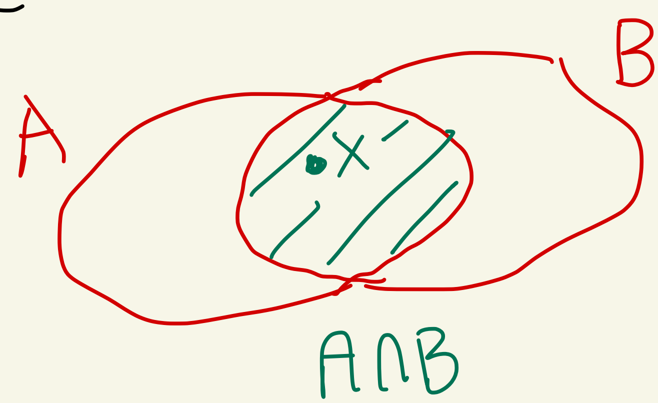
The <u>union</u> of A and B is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$



The <u>intersection</u> of A and B is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$



Ex:

$A = \{\sqrt{2}, 5, 9, 20, \frac{1}{3}, 2\}$

$B = \{-16, 9, \frac{1}{3}, 4\}$

$A \cup B = \{\underbrace{\sqrt{2}, 5, 9, 20, \frac{1}{3}, 2,}_{\text{A stuff}} \underbrace{-16, 4}_{\text{extra B stuff}}\}$

$A \cap B = \{9, \frac{1}{3}\}$

Method to show that $A = B$ when A and B are sets

1. Show that $A \subseteq B$
2. Show that $B \subseteq A$

Ex:

Let $A = \{2k \mid k \in \mathbb{Z}\}$
and $B = \{3n \mid n \in \mathbb{Z}\}$
Prove that $A \cap B = \{6\ell \mid \ell \in \mathbb{Z}\}$.

proof:

$\boxed{\subseteq}$: Let's show $A \cap B \subseteq \{6\ell \mid \ell \in \mathbb{Z}\}$.

Pick some $x \in A \cap B$.

Then, $x \in A$ and $x \in B$.

So, $x = 2k$ and $x = 3n$ where k, n are integers.

So, $2k = 3n$.

Thus, $3n$ is even. (because $3n$ is 2 times an integer)

We can't have n being odd since then $3n$ would be odd.
(odd * odd = odd)

So n is even.

Thus, $n = 2m$ where m is an integer.

So, $x = 3n = 3(2m) = 6m$

So, $x \in \{6\ell \mid \ell \in \mathbb{Z}\}$.

Thus, $A \cap B \subseteq \{6\ell \mid \ell \in \mathbb{Z}\}$.

$\boxed{\supseteq}$: Now let's show $\{6\ell \mid \ell \in \mathbb{Z}\} \subseteq A \cap B$.

Let $x \in \{6\ell \mid \ell \in \mathbb{Z}\}$.

Then $x = 6j$ where $j \in \mathbb{Z}$.

Thus, $x = 2(3j) \in A$.

And, $x = 3(2j) \in B$

So, $x \in A \cap B$.

Thus, $\{6\ell \mid \ell \in \mathbb{Z}\} \subseteq A \cap B$.

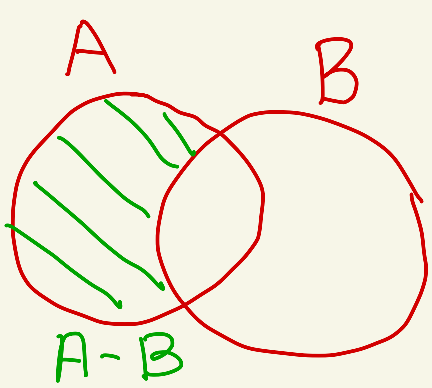
Def: Let A and B be sets. We say that A and B are <u>disjoint</u> if $A \cap B = \phi$ where ϕ is the empty set.

Ex: $A = \{1, 2\}$, $B = \{3, 4\}$

$A \cap B = \phi$

So, A and B are disjoint

Def: Let A and B be sets. The <u>difference</u> of A and B is



$$A - B = \underbrace{\{x \mid x \in A \text{ and } x \notin B\}}$$

<u>read:</u> "all x where x is in A and x is

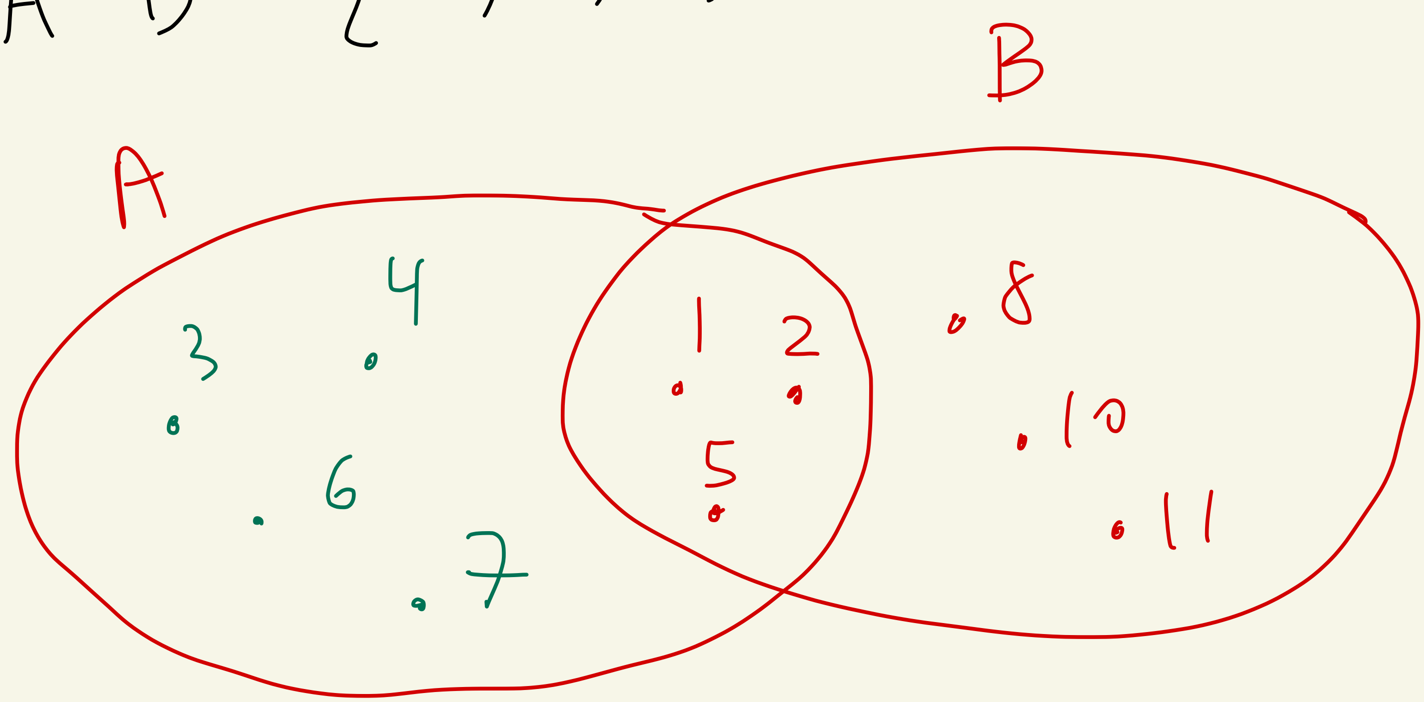
not in B"

Notation: Some people write $A \setminus B$ for $A - B$.

Ex: $A = \{1, 2, 3, 4, 5, 6, 7\}$

$B = \{8, 10, 11, 2, 5, 1\}$

$A - B = \{3, 4, 6, 7\}$



$B - A = \{8, 10, 11\}$

$A - \{10, 11, 20\} = A$

$A = \{1, 2, 3, 4, 5, 6, 7\}$

$A - A = \phi$ nothing left

HW problem

Let A, B, C be sets.

Prove: If $A \subseteq B$, then $A - C \subseteq B - C$

Proof: Suppose that $A \subseteq B$.

We want to show that $A - C \subseteq B - C$.

Let $x \in A - C$.

Then $x \in A$ and $x \notin C$.

Since $x \in A$ and $A \subseteq B$, we know that $x \in B$.

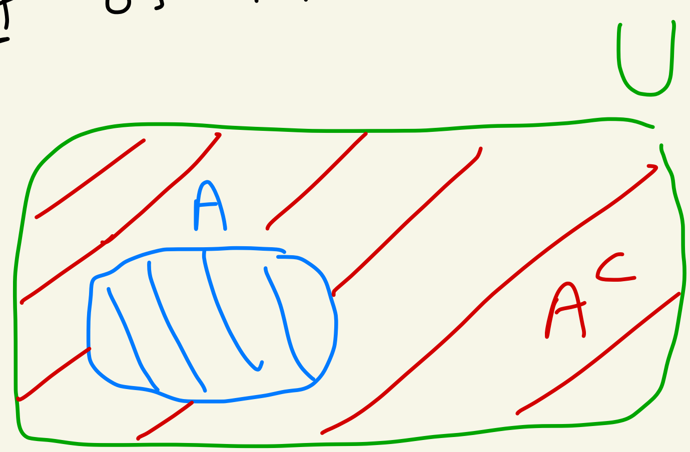
Thus, $x \in B$ and $x \notin C$.

Hence $x \in B - C$.

Therefore, $A - C \subseteq B - C$ ▨

- Sometimes all the sets you are looking at live inside of one big set. Let's call that big set a "universal set" or "universe"

Def: Let A be a set where U is a universal set (So, $A \subseteq U$) Then the complement of A with respect to U is



$$A^c = U - A$$
$$= \{x \mid x \in U \text{ and } x \notin A\}.$$

Ex:

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$A = \{2, 4, 6, 8, 10, 12\}$

$A^c = U - A = \{1, 3, 5, 7, 9, 11\}$

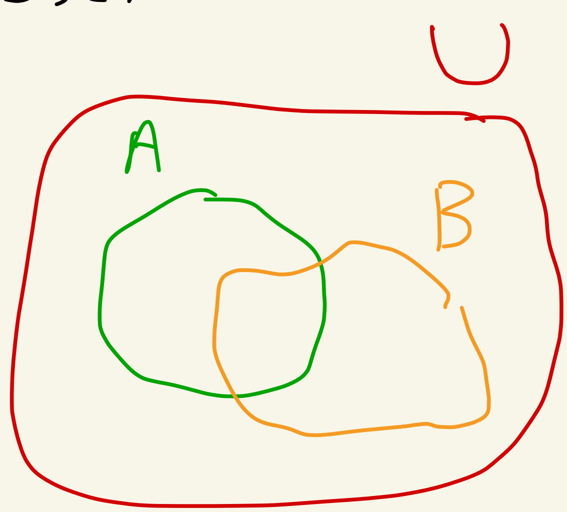
Theorem: (de Morgan's laws)

Let U be a universal set.
Let A and B be subsets of U.

Then:

① $(A \cup B)^c = A^c \cap B^c$

② $(A \cap B)^c = A^c \cup B^c$

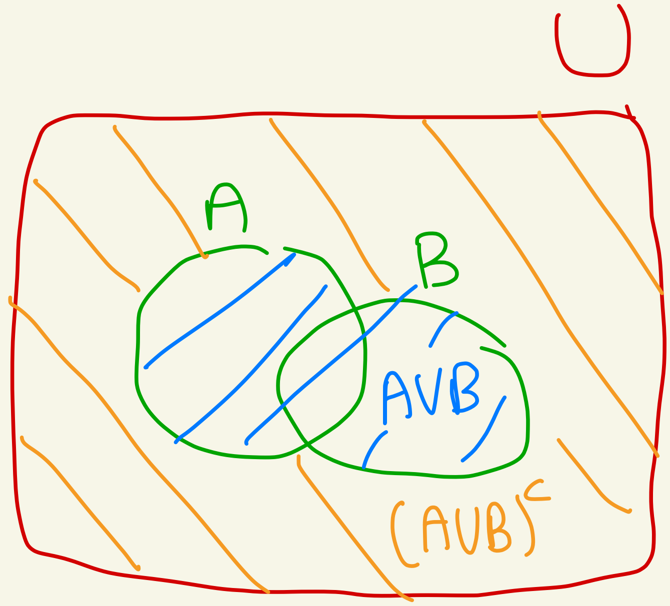


proof:

Let's prove 1. You can try 2.

$\boxed{\subseteq}$ Let's show $(A \cup B)^c \subseteq A^c \cap B^c$.

Let $x \in (A \cup B)^c$.



Then, $x \in U$
and $x \notin A \cup B$.

So, $x \in U$ and
"$x \in A \cup B$" is not true.

So, $x \in U$ and "$x \in A$ or $x \in B$" is not true.

So, $x \in U$ and $x \notin A$ and $x \notin B$. 2450

$\neg (P \text{ or } Q)$ equivalent
$(\neg P) \text{ and } (\neg Q)$
$\neg$ means not

Thus, $x \in A^c$
and $x \in B^c$.

So, $x \in A^c \cap B^c$.

Therefore, $(A \cup B)^c \subseteq A^c \cap B^c$.

$\boxed{\supseteq}$: Now let's show $A^c \cap B^c \subseteq (A \cup B)^c$.

Let $y \in A^c \cap B^c$.

So, $y \in A^c$ and $y \in B^c$.

So, $y \in U$ and $y \notin A$ and $y \notin B$. same logic as above

Thus, $y \in U$ and $y \notin A \cup B$

So, $y \in (A \cup B)^c$.

Thus, $A^c \cap B^c \subseteq (A \cup B)^c$.

By $\boxed{\subseteq}$ and $\boxed{\supseteq}$ we have $(A \cup B)^c = A^c \cap B^c$

$\blacksquare$

Another way to prove:

$x \in (A \cup B)^c$

iff $x \in U$ and $x \notin A \cup B$

iff $x \in U$ and $x \notin A$ and $x \notin B$

iff $x \in A^c$ and $x \in B^c$

iff $x \in A^c \cap B^c$.

Thus, $(A \cup B)^c = A^c \cap B^c$. $\blacksquare$

Def: Let A and B be sets. The Cartesian product of A and B is

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

Note: (a,b) is called an ordered pair. Order matters for (a,b)

People have proposed various set definitions for (a,b). For example one is $(a,b) = \{ a, \{a,b\} \}$

<u>Ex:</u> $A = \{1, 5, 9\}$

$B = \{4, 9\}$

$A \times B = \{(1,4), (1,9), (5,4), (5,9), (9,4), (9,9)\}$

$B \times B = \{(4,4), (4,9), (9,4), (9,9)\}$

<u>Note:</u> In general, if S and T are finite sets, then

$$\underbrace{|S \times T|}_{\text{means size of } S \times T} = \underbrace{|S|}_{\text{size of } S} \cdot \underbrace{|T|}_{\text{size of } T}$$

Ex:

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

$$\mathbb{Z} - \{0\} = \{\ldots, -4, -3, -2, -1, 1, 2, 3, 4, \ldots\}$$

Let $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$.

Then,

$$S = \{(a,b) \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$$

$$= \{(0,2), (-1,7), (3,-1), \ldots\}$$

infinitely many more

Ex: Let A, B, C be sets.
Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

proof:

$\boxed{\subseteq}$: We first show that
$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

Let $x \in A \times (B \cap C)$.
Then, $x = (m, n)$ where $m \in A$ and $n \in B \cap C$.
Since $n \in B \cap C$ we know $n \in B$ and $n \in C$.
Thus, $(m, n) \in A \times B$ since $m \in A$ and $n \in B$
and $(m, n) \in A \times C$ since $m \in A$ and $n \in C$.
Hence, $x = (m, n) \in (A \times B) \cap (A \times C)$.

$\boxed{\supseteq}$: Now we show that
$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

Let $y \in (A \times B) \cap (A \times C)$.
Then, $y \in A \times B$ and $y \in A \times C$.
So, $y = (o, p)$ where $o \in A$ and $p \in B$
and $p \in C$.

Since $p \in B$ and $p \in C$ we know $p \in B \cap C$.

Thus, $y = (\theta, p) \in A \times (B \cap C)$.

$\blacksquare$

__Def:__ Let A be a set. We define the __power set__ of A to be the set of all subsets of A, that is

$$\underbrace{\mathcal{P}(A)}_{\text{power set of } A} = \underbrace{\{B \mid B \subseteq A\}}_{\text{the set of all } B \text{ where } B \subseteq A}$$

Ex: $A = \{1, 2\}$

Subsets of A

$\emptyset$

$\{1\}$

$\{2\}$

$\{1, 2\}$

empty set is a subset of every set

$\emptyset = \{\ \}$

SIDE COMMENTARY

$S \subseteq T$ means:

$\forall x\,(\text{If } x \in S, \text{ then } x \in T)$

$\emptyset \subseteq T$ means:

$$(\forall x)\underbrace{(\text{If } \underbrace{x \in \emptyset}_{F}, \text{ then } x \in T)}_{T}$$

$$|\mathcal{P}(A)| = 4 = 2^2 = 2^{|A|}$$

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

Ex: $B = \{5, 2, 1\}$

$$\mathcal{P}(B) = \{ \emptyset, \{1\}, \{2\}, \{5\}, \{5,2\}, \{2,1\}, \{5,1\}, \{5,2,1\} \}$$

Note: $|\mathcal{P}(B)| = 8 = 2^3 = 2^{|B|}$

Theorem: If S is finite, then $|\mathcal{P}(S)| = 2^{|S|}$

Theorem: Let A and B be sets. Then, $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$.

proof:

($\Rightarrow$) It's clear that if $A = B$, then $\mathcal{P}(A) = \mathcal{P}(B)$.

($\Leftarrow$) Now we must prove "If $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$".

Suppose $\mathcal{P}(A) = \mathcal{P}(B)$.

To show that $A = B$ we

will show $A \subseteq B$ and $B \subseteq A$.

$\boxed{\text{Claim 1: } A \subseteq B}$

We know $A \subseteq A$.

So, $A \in \mathcal{P}(A)$.

Then, since $\mathcal{P}(A) = \mathcal{P}(B)$, we know $A \in \mathcal{P}(B)$.

Thus, $A \subseteq B$.

$\boxed{\text{claim 2: } B \subseteq A}$

You can do this proof the same way as claim 1, but let's change it up.

Let $b \in B$.

Then, $\{b\} \subseteq B$.

So, $\{b\} \in \mathcal{P}(B)$

Since $\mathcal{P}(B) = \mathcal{P}(A)$ we have $\{b\} \in \mathcal{P}(A)$

Thus, $\{b\} \subseteq A$.

Hence $b \in A$.

So, $B \subseteq A$.

By claim 1 and 2 we know $A = B$.

$\blacksquare$

Def: When every element of a set $\mathcal{A}$ is itself a set then we call $\mathcal{A}$ a <u>family</u> or <u>collection</u> of sets.

Ex: $\mathcal{P}(A)$ is a family of sets.

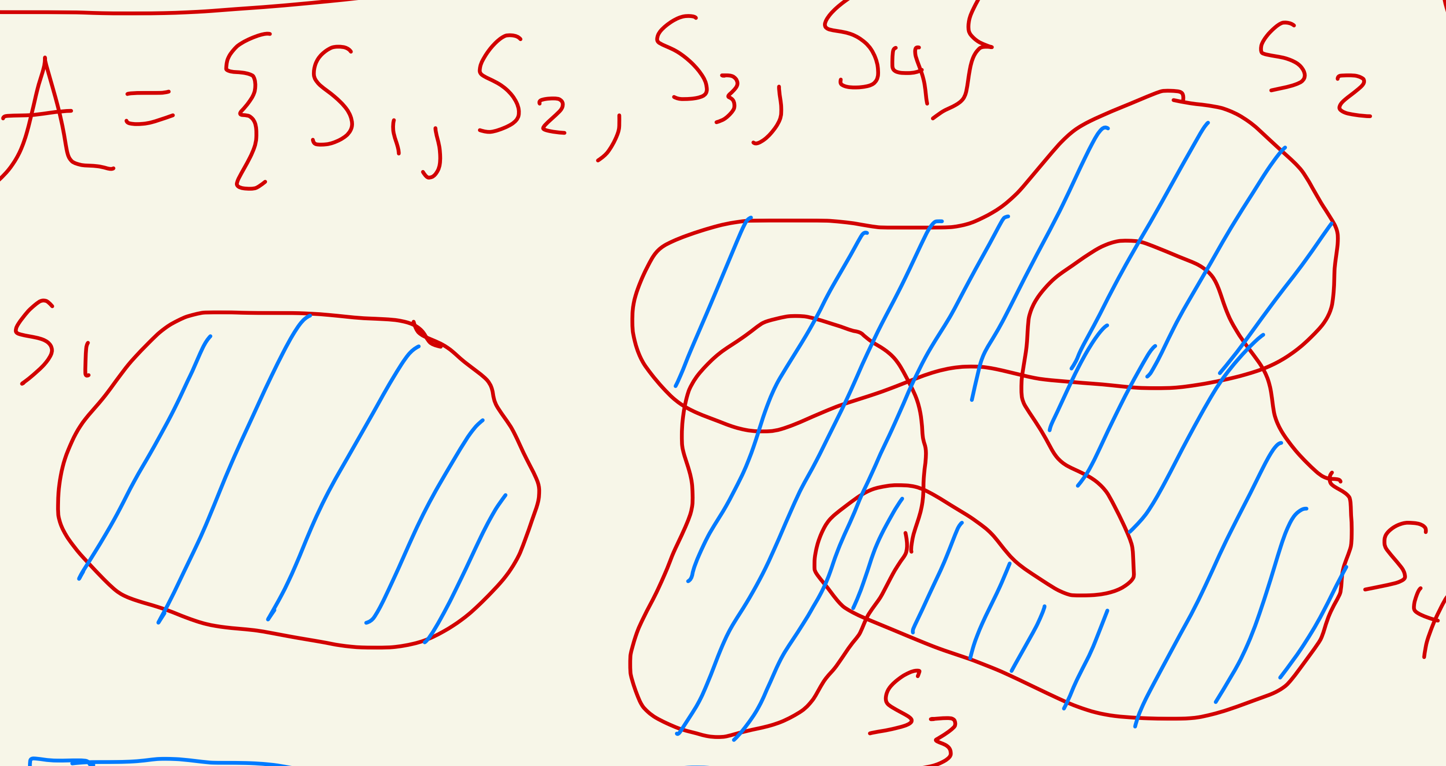
Def: Let $\mathcal{A}$ be a non-empty family of sets.

Define the union over $\mathcal{A}$ to be

$$\bigcup_{S \in \mathcal{A}} S = \{ x \mid x \in S \text{ for some } S \in \mathcal{A} \}$$
$$= \{ x \mid \text{there exists some } S \in \mathcal{A} \text{ where } x \in S \}$$

Ex:

$\mathcal{A} = \{ S_1, S_2, S_3, S_4 \}$

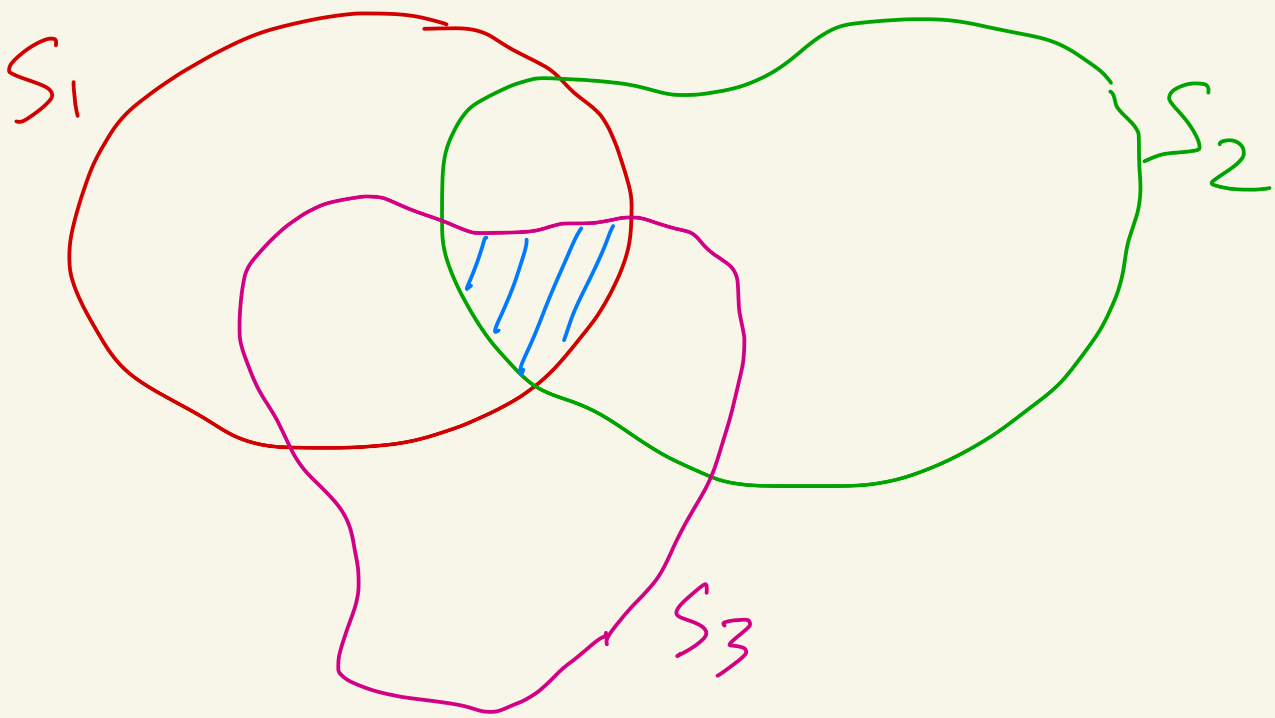


shaded blue is $\bigcup_{S \in \mathcal{A}} S$

Define the <u>intersection over $\mathcal{A}$</u> to be

$$\bigcap_{S \in \mathcal{A}} S = \{ x \mid x \in S \text{ for all } S \in \mathcal{A} \}$$

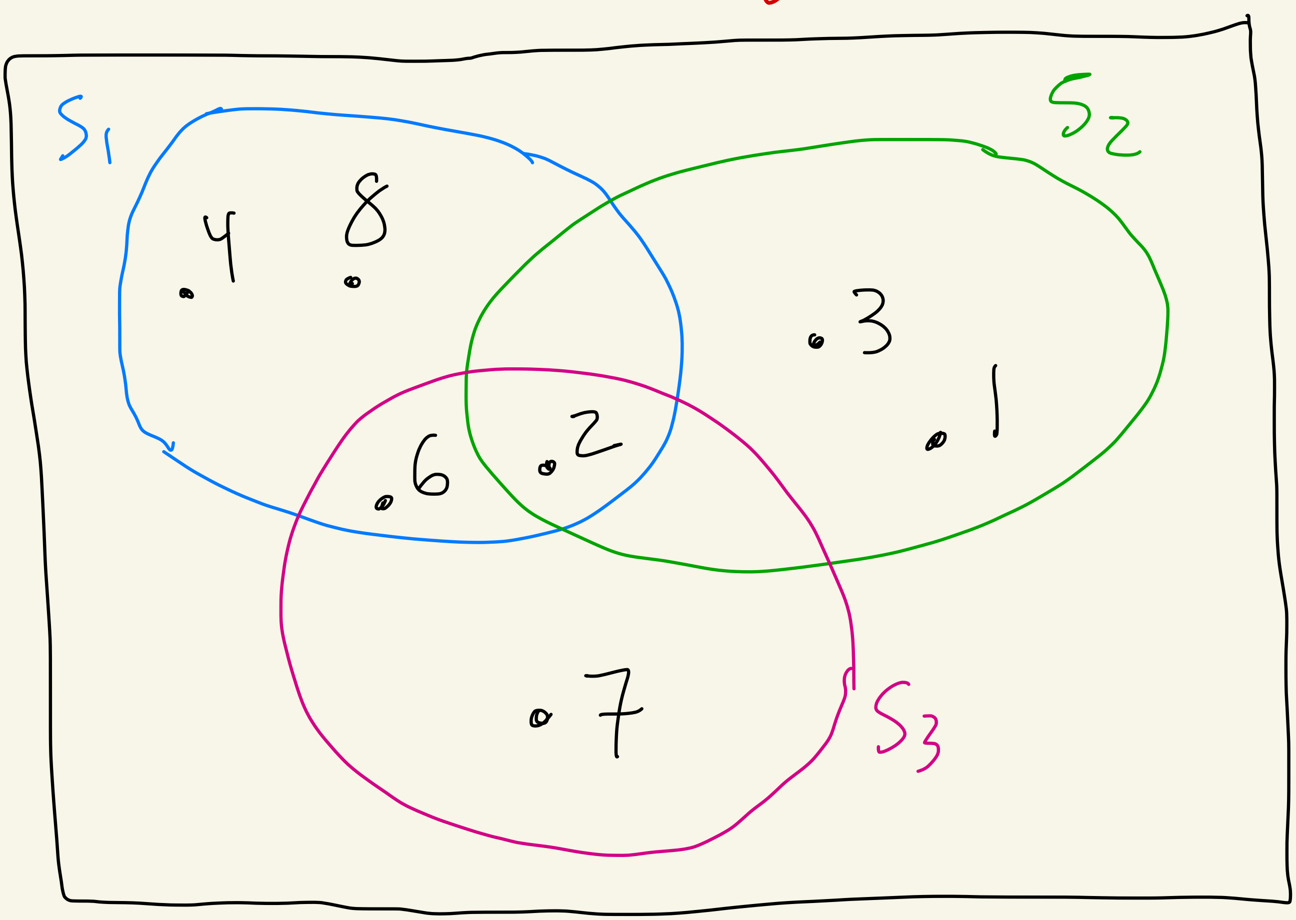
$\boxed{\text{Ex:}}$ $\mathcal{A} = \{ S_1, S_2, S_3 \}$



shaded blue is $\bigcap_{S \in \mathcal{A}} S$

Ex:

$$A = \left\{ \overbrace{\{2,4,6,8\}}^{S_1}, \overbrace{\{1,3,2\}}^{S_2}, \underbrace{\{2,7,6\}}_{S_3} \right\}$$



$$\bigcup_{S\in A} S = \{1,2,3,4,6,7,8\}$$

$$\bigcap_{S\in A} S = \{2\}$$

$\boxed{\text{Ex:}}$ $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

$\mathbb{Z}, \{\dots, -2, -1, 0, 1, 2, \dots\}$

$$\mathcal{B} = \{ \overbrace{\{n \in \mathbb{Z} \mid |n| \leq k\}}^{S_k} \mid k \in \mathbb{N} \}$$

$$= \{ S_k \mid k \in \mathbb{N} \}$$

$$= \{ S_1, S_2, S_3, S_4, \dots \}$$

and $S_k = \{n \in \mathbb{Z} \mid |n| \leq k\}$

$$S_1 = \{n \in \mathbb{Z} \mid |n| \leq 1\} = \{-1, 0, 1\}$$

$$S_2 = \{n \in \mathbb{Z} \mid |n| \leq 2\} = \{-2, -1, 0, 1, 2\}$$

$$S_3 = \{-3, -2, -1, 0, 1, 2, 3\}$$

$S_4 = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

And so on...

$$\bigcup_{S \in B} S = \mathbb{Z}$$

$\boxed{\bigcup_{k=1}^{\infty} S_k \text{ is another way to write it}}$

$$\bigcap_{S \in B} S = \{-1, 0, 1\}$$

$\boxed{\bigcap_{k=1}^{\infty} S_k \text{ is another way to write it}}$

<u>Def</u>: Let I be a non-empty set. Suppose for each $\alpha \in I$ there is a corresponding set A_α.
The family

$$\mathcal{A} = \{ A_\alpha \mid \alpha \in I \}$$

is called an <u>indexed family of sets</u>. The set I is called the <u>index set</u>.
If $\alpha \in I$, then α is called the <u>index</u> of A_α.

Ex: Previously we had

$\mathcal{B} = \{ S_k \mid k \in \mathbb{N} \}$

Here $\mathcal{B}$ is an indexed family of sets. $\mathbb{N}$ is the index set.

S_3 ← $\alpha = 3$ is the index of S_3

<u>Def:</u> Let I be a non-empty set. Let $\mathcal{A} = \{A_\alpha \mid \alpha \in I\}$ be an indexed family of sets.

Define the <u>union over $\mathcal{A}$</u>

$$\bigcup_{\alpha \in I} A_\alpha = \left\{ x \;\middle|\; \begin{array}{l} \text{there exists } \alpha \in I \\ \text{with } x \in A_\alpha \end{array} \right\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \left\{ x \;\middle|\; \begin{array}{c} x \in A_\alpha \text{ for} \\ \text{all } \alpha \in I \end{array} \right\}$$

$\boxed{\text{Ex:}}$ In our previous example $\mathcal{B} = \{S_k \mid k \in \mathbb{N}\}$

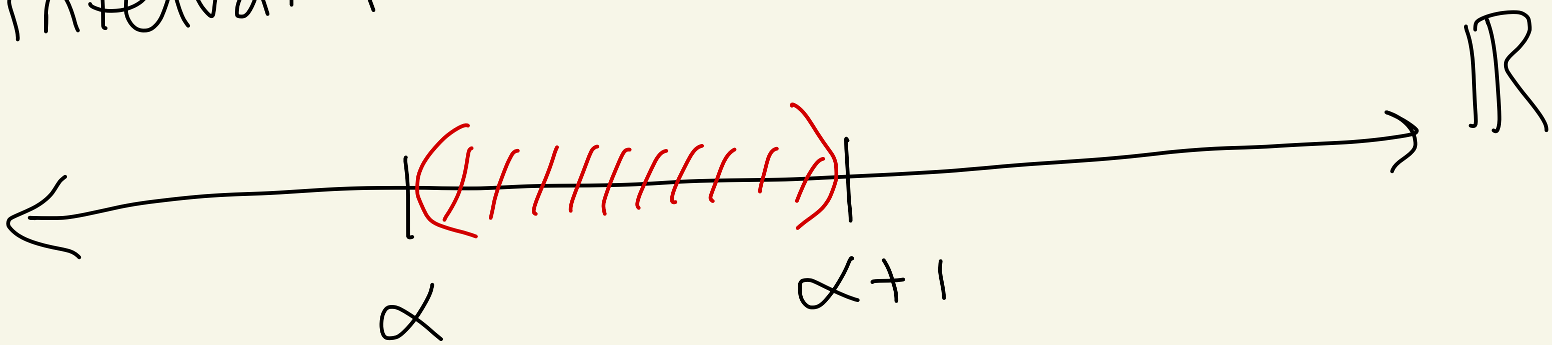
we would write

$$\bigcup_{k \in \mathbb{N}} S_k = \mathbb{Z} \quad \text{and} \quad \bigcap_{k \in \mathbb{N}} S_k = \{-1, 0, 1\}$$

$\boxed{\text{Ex:}}$ Let's make sense of

$$\bigcup_{\alpha \in \mathbb{Z}} (\alpha, \alpha+1) \quad \text{and} \quad \bigcap_{\alpha \in \mathbb{Z}} (\alpha, \alpha+1)$$

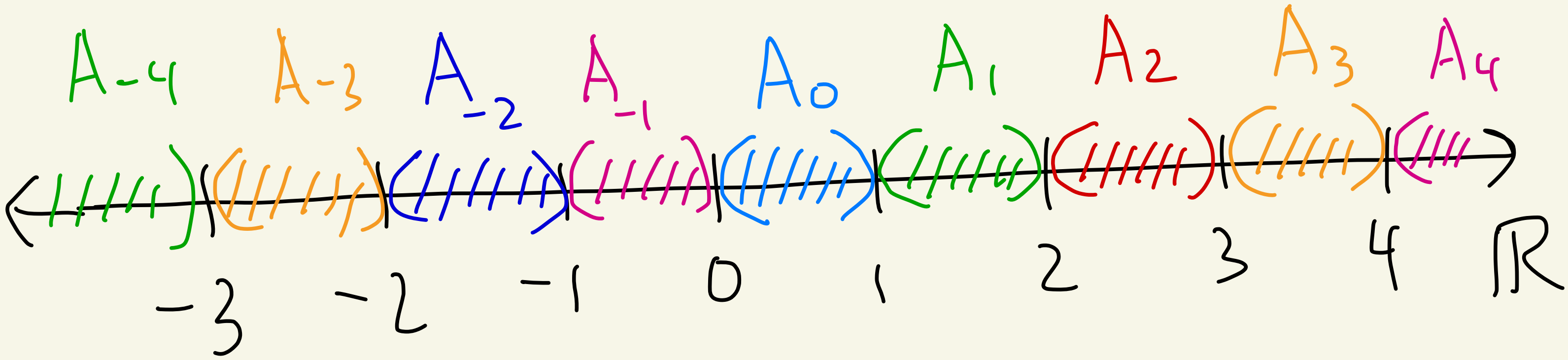
where $(\alpha, \alpha+1)$ means the interval in the real numbers $\mathbb{R}$



$I = \mathbb{Z}$

$A_\alpha = (\alpha, \alpha+1)$

$\mathcal{A} = \{A_\alpha \mid \alpha \in \mathbb{Z}\}$



$$\underbrace{\bigcup_{\alpha \in \mathbb{Z}} A_\alpha}_{} = \mathbb{R} - \mathbb{Z}$$

another way to write this is $\bigcup_{\alpha=-\infty}^{\infty} A_\alpha$

its understood α ranges over whole numbers $\mathbb{Z}$

$$\underbrace{\bigcap_{\alpha \in \mathbb{Z}} A_\alpha}_{} = \phi$$

another way to write $\bigcap_{\alpha=-\infty}^{\infty} A_\alpha$

its understood α ranges over whole numbers $\mathbb{Z}$

Theorem: Let $\mathcal{A} = \{A_\alpha \mid \alpha \in I\}$ be an indexed family of sets. Let $\alpha_0 \in I$ be a fixed element.

Then:

① $A_{\alpha_0} \subseteq \bigcup_{\alpha \in I} A_\alpha$

② $\bigcap_{\alpha \in I} A_\alpha \subseteq A_{\alpha_0}$

Ex:

$I = \mathbb{N}$

$\alpha_0 = 5$

$A_5 \subseteq \bigcup_{\alpha \in \mathbb{N}} A_\alpha$

$\bigcap_{\alpha \in \mathbb{N}} A_\alpha \subseteq A_5$

proof:

① Pick some $x \in A_{\alpha_0}$.

Then there exists $\alpha \in I$ (namely $\alpha = \alpha_0$) where $x \in A_\alpha$

Thus,

$$x \in \bigcup_{\alpha \in I} A_\alpha = \left\{ y \;\middle|\; \begin{array}{c} \text{there exists} \\ \alpha \in I \text{ with} \\ y \in A_\alpha \end{array} \right\}$$

Therefore,

$$A_{\alpha_0} \subseteq \bigcup_{\alpha \in I} A_\alpha$$

2 You try. $\blacksquare$